

Network source location by entropic message passing

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The ground state entropy of network source location problems is derived at both the replica symmetric level and one-step replica symmetry breaking level using the entropic cavity method. The recursive relations for the cavity probability and entropy are obtained in a probabilistic way by focusing on the change of the ground state size under the cavity iterations. The resulting entropic message passing inspired decimation and reinforcement algorithms identify the optimal location of sources in single instances of transportation networks. The traditional belief propagation without taking the entropic effect into account is also compared. We find that in the glassy phase the reinforcement algorithm is fastest while the entropic message passing inspired decimation yields a value of the fraction of source nodes in the optimized network closest to the ground state value.

Keywords: cavity and replica method, disordered systems (theory), message-passing algorithms, communication, supply and information networks

I. INTRODUCTION

Statistical physics methods and ideas inherited from studies of disorder systems [1, 2] play an important role in providing theoretical insights and developing low complexity algorithms for combinatorial optimization problems or constraint satisfaction problems. These problems become subjects of interest across a variety of different disciplines such as computer science, discrete mathematics, statistical physics, engineering and computational biology [3]. One archetype that involves both continuous and discrete variables is the network source location problem where we try to find optimal locations of sources in transportation networks with an optimal flow pattern. Studies on this source location problem [4–7] are practically relevant to network design and optimization [8] and can have widespread applications in the field of operations research [9].

Transportation networks are composed of nodes with either a surplus or a deficiency of resources. How to distribute the resources and to choose some (deficient) nodes for resource provisions becomes an important problem in network optimization. In the source location problem, we are given a random distribution of resource providing nodes (surplus nodes) in a network and the objective is to optimize the actual location of additional source nodes so as to achieve the minimal installation cost of the new source providers as well as the corresponding transportation cost [7]. After the optimization, the remaining deficient nodes serve as consumer nodes and a network-wide satisfaction is achieved. Previous studies showed that the cavity energy functions with continuous variables can be decomposed into composite functions parameterized by their energy minima, such that the recursions of cavity energy functions can be converted to simple recursions of probabilities, which simplifies the analysis a lot [4, 6, 7, 10]. The resulting picture is a cascade of phase transitions in the glassy phase and the fraction of source nodes in the final optimized network shows discontinuous

transitions with increasing installation cost and this implies that different configurations of the source and consumer nodes will become energetically stable as the installation cost varies [5, 7]. Two of them are the singlet regime where the consumer nodes are isolated and the doublet regime where the consumer nodes can be paired or isolated. Even in the singlet regime, there exists glassy phase when the fraction of surplus nodes is below 0.25 for a random regular network with node degree equal to 3 [7]. In the glassy phase, optimizing the locations of sources becomes algorithmically hard, which is most likely due to the emergence of long range frustration [11, 12]. Ref. [7] revealed that a random assignment of the bistable node with the same cavity energy in the consumer state or source state typically causes contradictions throughout the network. That is to say, the unfrozen nodes are strongly correlated in this region. Therefore, for those bistable nodes, we should consider their entropic effects and we expect that the entropic information extracted from the ground state can help to infer the correct assignment for each node.

In this work, we assume that the cavity iterations [2], i.e., adding one node or one link between two nodes into the original system, will affect the size of the ground state (the number of optimal assignments or solutions), such that the recursive relations for the cavity probability and entropy can be derived in a probabilistic way. Hereafter, we identify an assignment to be optimal if it yields the minimal energy cost in a specific regime (e.g., the singlet or doublet connection pattern). This entropic cavity method has been used to compute the ground state entropy of the minimal vertex cover problem [13]. Similar studies on the channel coding problem and budget-constrained auctions can be found in Refs. [14, 15]. We emphasize here that the bistable node can either be in the source state or consumer state since both states share the same energy [7], however, taking the entropic contribution into account, these two states may not have equal probabilities. Nonetheless, this entropic effect was ignored in previous studies [4–7]. Furthermore, by taking the zero temperature limit simply, the entropy term has vanishing contributions to the free energy and the ground state entropy can not be considered properly. In our current work, we shall demonstrate that not only can the entropic effect be used to evaluate the typical value of the ground state entropy as a function of the fraction of the surplus nodes (quenched disorder), but also can be used for algorithmic purposes on single random instances. We apply two kinds of decimation strategy to identify the optimal location of sources. One is the maximal decimation in which the most polarized nodes are fixed once the algorithm converges and then the network is simplified correspondingly. This strategy can be thought as a hard decimation since it is equivalent to adding an external field of infinite intensity to the decimated nodes [16–18]. The other is the reinforcement strategy which can be viewed as a sort of soft decimation. In this strategy, the cavity probability is strengthened or attenuated at each step by an external bias whose intensity is updated with a rate increasing with the run time [19–21]. When the updating rate and the intensity of the external bias are correctly chosen, the external messages are able to drive the iterations towards some optimal assignment thanks to the fact that this soft decimation utilizes the global information of the network at each step. We find that in terms of computation complexity the reinforcement strategy is much more efficient than the hard decimation in the glassy phase. However, the entropic message passing inspired decimation is able to yield a value of the fraction of source nodes in the optimized network closest to the ground state value.

The rest of this paper is organized as follows. The source location problem is defined in more detail in Sec. II. The analysis at both the replica symmetric (RS) level and one-step replica symmetry broken (1RSB) level by the entropic cavity method is presented in Sec. III.

In Sec. IV, experimental studies of the proposed maximal decimation and reinforcement strategy are carried out. Conclusion and some future directions are given in Sec. V.

II. THE SOURCE LOCATION PROBLEM

We consider a transportation network of N nodes in which resources are transported through the links, so that the resource demands of all nodes are satisfied. There are two kinds of nodes. The *surplus nodes* supply the resources and the *deficient nodes* have demands for resources. Respectively, ϕ_s and $1 - \phi_s$ represent the fractions of surplus and deficient nodes in the network. The amount of resources supplied or demanded by a node i ($i = 1, \dots, N$) is described by its capacity Λ_i being positive and negative for surplus and deficient nodes respectively. To avoid considerations imposed by capacity limitations, we consider the generic case where the capacities of all deficient nodes are uniform and equal to -1 and those of all surplus nodes are uniform and equal to a sufficiently large real number A , so that resources from the surplus nodes are distributed to their neighbors with an optimal traffic pattern independent of the details of the capacity distribution. This scenario is typical in many modern applications of network traffic optimization, such as logistic networks and sensor networks, where the surplus nodes represent distribution centers in logistic networks or base stations in sensor networks. The locations of the surplus and deficient nodes are random, hence the capacity distribution of a node is given by $\rho(\Lambda) = \phi_s \delta(\Lambda - A) + (1 - \phi_s) \delta(\Lambda + 1)$. The network structure considered in this paper is a diluted network in which each node has the same degree C but the analysis below can be easily extended to the network with fluctuating degree (e.g., Erdős-Rényi random network).

The source location problem is an important problem in dynamic network management. The task is to identify the locations of an appropriate number of deficient nodes and convert them to resource providers with capacity $A \gg 1$; these converted nodes are referred to as *source nodes* (the source nodes also include the surplus nodes when we consider the fraction of the source nodes in the final optimized network), whereas those nodes that remain as resource consumers are referred to as *consumer nodes*. The purpose is to reduce the overall transportation cost. On the other hand, the installation of source nodes has an installation cost, so that the total cost function is given by

$$E = \sum_{(ij)} \frac{x_{ij}^2}{2} + \frac{u^2}{2} \sum_{i \in \mathcal{D}} s_i. \quad (1)$$

The first term is the total transportation cost, summed over the transportation cost of each link (ij) quadratic in the current $x_{ij} = -x_{ji}$ which denotes the real-valued flow from node j to i . The quadratic function is chosen because of its convex property, which tends to balance the traffic load among the links; other convex functions can work equally well [22–24]. The flows are required to satisfy the non-negativity constraints of the final resources given by

$$\xi_i \equiv \Lambda_i + \sum_{j \in \partial i} x_{ij} \geq 0, \quad (2)$$

where ξ_i is the final resource of node i and ∂i denotes neighbors of node i .

In the second term, $u^2/2$ is the installation cost of a source node, where u has the same dimension as the flows. \mathcal{D} denotes the set of deficient nodes, and $s_i = 1$ if node i is assigned

to be a source node while 0 for being a consumer node. Hence the source location problem is an optimization problem in the space of real-valued variables x_{ij} and the Boolean variables s_i .

It was found in Ref. [7] that the optimal choice of the source nodes can be obtained by assuming that all deficient nodes are consumer nodes, and then optimizing the nonlinear cost function in the space of the real-valued variables x_{ij} given by [5]

$$E = \sum_{(ij)} \frac{x_{ij}^2}{2} + \frac{u^2}{2} \sum_{i \in \mathcal{D}} \Theta(-\xi_i), \quad (3)$$

where $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ otherwise. Then we identify those nodes with non-negative final resources as the consumer nodes, and assign those nodes with negative final resources to be the source nodes in the optimal solution of the source location problem.

As the installation cost parameter u changes, different configurations of consumer and source nodes appear [5, 7]. This leads to a cascade of phase transitions happen in the glassy phase with abrupt jumps of the fraction of source nodes in the optimized network. For instance, in the singlet regime where $1/\sqrt{C} < u < \sqrt{(C+1)/[C(C-1)]}$, the consumer nodes are isolated. This is because in this range, the singly consuming state is always energetically more stable compared with its two neighboring phases such as all-source phase and doublet phase [7]. The simple flow configuration in the singlet phase allows us to determine the optimal solution easily. The flow on a link into a consumer node is always $1/C$, and the flows on other links are 0 [7]. Hence the flow configuration is determined once the nodes are determined to be in the source state or the consumer state. This further simplifies the source location problem into a discrete-valued optimization problem.

Based on the cavity method, a comparison between the energies of the cavity source state and the cavity consumer state led to a belief propagation algorithm [5, 7]. When the fraction of surplus nodes is sufficiently large, it converges satisfactorily and provides excellent agreement with the simulation results in terms of the fraction of source nodes [7]. However, the algorithm was less satisfactory when the fraction of surplus nodes is not large. This is the regime where the cavity recursive relations become unstable towards fluctuations or, in the framework of the replica method, where the replica symmetric solution becomes unstable.

One drawback of the previous analysis is that the degeneracy of the cavity source state and the cavity consumer state has been ignored. This was evident in Ref. [7] in which the cavity bistable state and the cavity source state were grouped together in deriving their recursion relations. Thus the entropy of the solution space was not considered properly. In this paper, we will compute the entropy of the ground state extending the previous works [7, 13], restricting our attention to the singlet regime. We will also propose the entropic message passing inspired decimation algorithm and reinforcement strategy to identify the optimal location of sources in both the easy (replica symmetric) and hard (replica symmetry breaking) phases.

III. THE ENTROPIC CAVITY METHOD

In this section, we will present the entropic cavity method to compute the ground state entropy of the source location problem at the replica symmetric level and one-step replica symmetry breaking level. At the replica symmetric level, there exists a single ground state

and the clustering hypothesis that the correlation between any two randomly selected nodes is weak becomes valid due to the sparse structure of the network. Following the work of Ref. [13], we focus on the change of the ground state size under the cavity iterations, and explicitly derive a closed set of equations involving two messages (cavity probability and cavity entropy). The replica symmetric ground state entropy can be evaluated from the fixed point of the recursive equations. Extending the analysis to one-step replica symmetry broken level is straightforward. When replica symmetry is broken, the single ground state would split into exponentially many states, which violates the clustering hypothesis. However, the weak correlation assumption is still satisfied in each state but the energy level crossings of states under the cavity iterations should be taken into account. All mean-field analysis presented here are restricted to the zero temperature limit which selects the ground state.

A. Replica symmetric analysis

Under the replica symmetric approximation, the joint state distribution of any two randomly chosen nodes from the large diluted network takes a factorized form making the derivation of a recursive relation feasible. Applying the cavity method to the source location problem, we consider the state of a node i in the absence of one of its neighbors j . In the singlet regime, it can be either in the cavity consumer state, or the cavity source state. In the cavity consumer state, the cavity energy of a node is lowest when a flow of $1/C$ enters it, whereas in the cavity source state, the cavity energy of the node is lowest when no flow enters it.

Let us first define $\psi_{i \rightarrow j}^s$ as the cavity probability that node i is in a source state in the absence of node j . It can also be viewed as the message passing from the node i to node j . If $\psi_{i \rightarrow j}^s = 1$, we say that node i is frozen to source state in the absence of node j . $\psi_{i \rightarrow j}^s = 0$ indicates node i should take the consumer state without node j . Otherwise, if $\psi_{i \rightarrow j}^s \in (0, 1)$, then the source and consumer states of node i are degenerate without node j . The value for the cavity probability $\psi_{i \rightarrow j}^s$ of node i depends on the incoming cavity probabilities from its neighbors other than node j , which can be categorized into three cases [7].

In the first case as depicted in figure 1 (a), all neighbors of node i have non-zero cavity probabilities $\{\psi_{k \rightarrow i}^s\}$. In this case, the cavity state of node i must be a consumer. We assume that the number of optimal assignments before addition of node i is Ω_{N-1} where N is the number of nodes in the network. After the node addition in figure 1 (a), Ω_{N-1} should be reduced since node i is now in the consumer state and should remain as a singlet. Hence all its neighbors besides j should take source state and only those configurations with $s_k = 1 (k \in \partial i \setminus j)$ in the ground state of the system with $N - 1$ nodes are valid after the addition of node i . $\partial i \setminus j$ denotes the neighbors of node i except node j . In this case, $\psi_{i \rightarrow j}^s = 0$, and the number of optimal assignments $\Omega_N = \Omega_{N-1} \prod_{k \in \partial i \setminus j} \psi_{k \rightarrow i}^s$ where the product comes from the weak correlation assumption. The cavity entropy change is readily obtained as

$$\Delta S_{i \rightarrow j} = \log \frac{\Omega_N}{\Omega_{N-1}} = \sum_{k \in \partial i \setminus j} \log \psi_{k \rightarrow i}^s. \quad (4)$$

The second case (figure 1 (b)) gets a bit more involved. In this case, only one neighbor of node i , say node k , is frozen to the consumer state in the absence of node i , i.e., $\psi_{k \rightarrow i}^s = 0$.

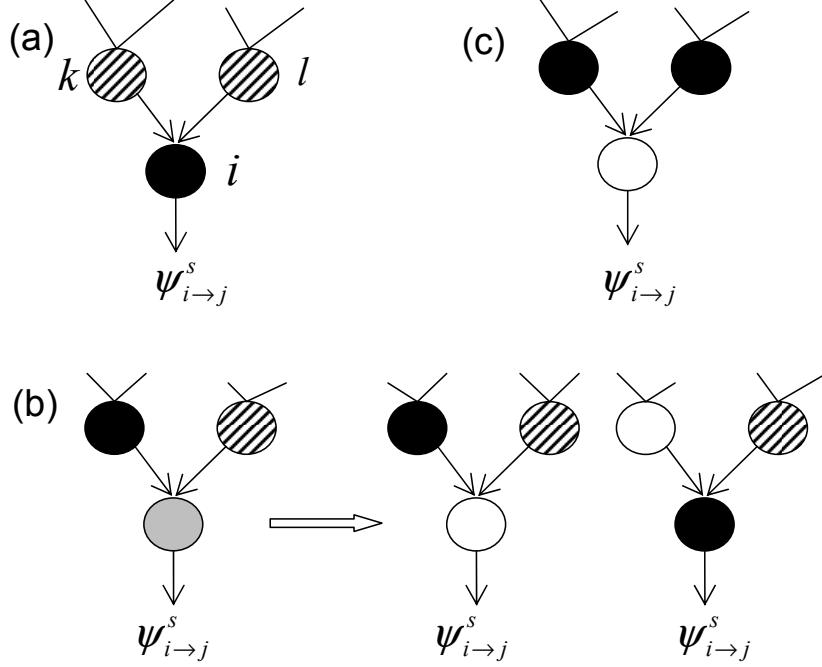


FIG. 1: Entropic contribution due to node addition (adding node i and its adjacent links). Each node is connected to $C = 3$ nodes. The codes for the cavity probabilities are: shaded for $\psi_{k \rightarrow i}^s \in (0, 1]$, black for $\psi_{k \rightarrow i}^s = 0$, white for $\psi_{k \rightarrow i}^s = 1$ and gray for $\psi_{k \rightarrow i}^s \in (0, 1)$. The arrow shows the message passing direction. (a) None of the neighbors of node i is frozen to the consumer state in the absence of node i . (b) Only one of the neighbors of node i , say node k , is frozen to the consumer state without node i . After the addition of node i , it can take the source state or consumer state. (c) At least two of neighbors of node i are frozen to the consumer state in the absence of node i .

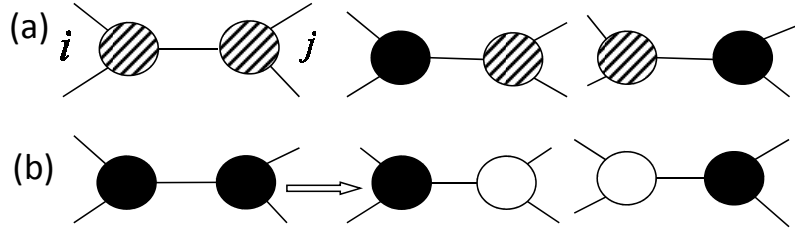


FIG. 2: Entropic contribution due to edge addition. The probabilistic meaning for each node is the same as that in figure 1. (a) At most one end of the added link is frozen into consumer state before the link addition. (b) Both ends of the added link are frozen into the consumer state in the absence of the link. Then either node i or j should change its state after the link is added.

According to figure 1 (b), the outcome is that the cavity source and consumer states of node i are degenerate. In Ref. [7], node i in this case was treated as the so-called bistable node. To consider the entropy change, we note that before the addition of node i , the number of optimal assignments in the ground state is $\Omega_{N-1} = \Omega_{N-2} \prod_{k' \in \partial k \setminus i} \psi_{k' \rightarrow k}^s$ where Ω_{N-2} is the number of optimal assignments in the ground state of the network without node k and i . Note that when node k was frozen to the consumer state, the same constraint

as the first case, namely, that $\psi_{k' \rightarrow k}^s > 0$ for all $k' \in \partial k \setminus i$, was already imposed on node k . Following the discussion on the first case, this also implies that Ω_{N-1} can be simplified to $\Omega_{N-1} = \Omega_{N-2} e^{\Delta S_{k \rightarrow i}}$ where $\Delta S_{k \rightarrow i} = \sum_{k' \in \partial k \setminus i} \log \psi_{k' \rightarrow k}^s$. The following two paragraphs analyze separately the possibilities that node i takes the source or consumer states.

If node i takes source state as shown in the middle panel of figure 1 (b), node k needs not change its state, therefore the number of optimal assignments after node addition with $s_i = 1$ is the same as Ω_{N-1} . Hence we have $\Omega_N|_{s_i=1} = \Omega_{N-2} e^{\Delta S_{k \rightarrow i}}$.

However, if node i chooses the consumer state, then the node k should change its state from consumer to source since we focus on the singlet regime where no paired consumer nodes are allowed. This change does not impose any further restrictions on the set of its neighbors $\partial k \setminus i$, since the neighbors of a source node can either be sources or consumers. On the other hand, assigning node i to be in the consumer state restricts all its neighbors besides k to be in source states only. Consequently, the number of optimal assignments after node addition with $s_i = 0$ is $\Omega_N|_{s_i=0} = \Omega_{N-2} \prod_{l \in \partial i \setminus k, j} \psi_{l \rightarrow i}^s$ where node i is fixed to consumer state (see the right panel of figure 1 (b)).

Since node i has the choices to be in those ground state assignments that have $s_i = 0$ or $s_i = 1$, the total number of optimal assignments after addition of node i is $\Omega_N = \Omega_{N-2} e^{\Delta S_{k \rightarrow i}} + \Omega_{N-2} \prod_{l \in \partial i \setminus k, j} \psi_{l \rightarrow i}^s$. The associated entropy change can be expressed as

$$\Delta S_{i \rightarrow j} = \log \frac{\Omega_N}{\Omega_{N-1}} = \log \left[1 + e^{-\Delta S_{k \rightarrow i}} \prod_{l \in \partial i \setminus k, j} \psi_{l \rightarrow i}^s \right]. \quad (5)$$

At the same time, the cavity probability $\psi_{i \rightarrow j}^s$ is determined by

$$\begin{aligned} \psi_{i \rightarrow j}^s &= \frac{\Omega_N|_{s_i=1}}{\Omega_N|_{s_i=1} + \Omega_N|_{s_i=0}} \\ &= \frac{1}{1 + e^{-\Delta S_{k \rightarrow i}} \prod_{l \in \partial i \setminus k, j} \psi_{l \rightarrow i}^s}. \end{aligned} \quad (6)$$

The third case where at least two of incoming $\psi_{k' \rightarrow i}^s$ for node i vanish is presented in figure 1 (c). The added node i should take the source state [7], i.e., $\psi_{i \rightarrow j}^s = 1$. In figure 1 (c), both $\psi_{k \rightarrow i}^s$ and $\psi_{l \rightarrow i}^s$ are equal to zero, thus the number of optimal assignments before the node addition is $\Omega_{N-1} = \Omega_{N-3} \prod_{k' \in \partial k \setminus i} \psi_{k' \rightarrow k}^s \prod_{l' \in \partial l \setminus i} \psi_{l' \rightarrow l}^s$ where Ω_{N-3} is the number of optimal assignments in the ground state of the network without node k, l and i . After node i is added, node i is frozen to source state and the neighbors k and l need not change their states, as a result, the number of optimal assignments Ω_N after addition of node i is identical to Ω_{N-1} . We conclude that the cavity entropy change for the third case is zero.

Collecting these results, we can write the entropic message passing equations in a compact

form as

$$\begin{aligned} \psi_{i \rightarrow j}^s &= 1 - \Theta \left(\prod_{k \in \partial i \setminus j} \psi_{k \rightarrow i}^s \right) + \sum_{k \in \partial i \setminus j} [1 - \Theta(\psi_{k \rightarrow i}^s)] \Theta \left(\prod_{l \in \partial i \setminus j, k} \psi_{l \rightarrow i}^s \right) \\ &\times \left[\frac{1}{1 + e^{-\Delta S_{k \rightarrow i}} \prod_{l \in \partial i \setminus k, j} \psi_{l \rightarrow i}^s} - 1 \right], \end{aligned} \quad (7a)$$

$$\begin{aligned} \Delta S_{i \rightarrow j} &= \Theta \left(\prod_{k \in \partial i \setminus j} \psi_{k \rightarrow i}^s \right) \sum_{k \in \partial i \setminus j} \log \psi_{k \rightarrow i}^s + \sum_{k \in \partial i \setminus j} [1 - \Theta(\psi_{k \rightarrow i}^s)] \Theta \left(\prod_{l \in \partial i \setminus j, k} \psi_{l \rightarrow i}^s \right) \\ &\times \log \left[1 + e^{-\Delta S_{k \rightarrow i}} \prod_{l \in \partial i \setminus k, j} \psi_{l \rightarrow i}^s \right], \end{aligned} \quad (7b)$$

where a pair of messages (cavity probability and cavity entropy) are involved. In the above analysis, the added node is assumed to be deficient node. However, a finite fraction of surplus nodes with very large capacities are present in the transportation network as the quenched disorder. The addition of a surplus node is assumed to have no entropy contribution to the network and its full and cavity probabilities are always fixed to be 1 since it is frozen to the source state by definition. Adding a surplus node will yield different cavity energies depending on the states of its neighbors since the consumer neighbor will draw resources from its adjacent surplus node [7]. This leads to the recursive relations $\psi_{i \rightarrow j}^s = \delta_{\Lambda_i, A} + \psi_{i \rightarrow j}^s|_{\Lambda_i=-1}$ and $\Delta S_{i \rightarrow j} = \delta_{\Lambda_i, -1} \Delta S_{i \rightarrow j}|_{\Lambda_i=-1}$, where $\psi_{i \rightarrow j}^s|_{\Lambda_i=-1}$ and $\Delta S_{i \rightarrow j}|_{\Lambda_i=-1}$ are given by Eqs. (7a) and (7b) respectively. The entropy change ΔS_i of adding node i can be computed using Eq. (7b) where $\partial i \setminus j$ is replaced by ∂i as

$$\begin{aligned} \Delta S_i &= \Theta \left(\prod_{k \in \partial i} \psi_{k \rightarrow i}^s \right) \sum_{k \in \partial i} \log \psi_{k \rightarrow i}^s + \sum_{k \in \partial i} [1 - \Theta(\psi_{k \rightarrow i}^s)] \Theta \left(\prod_{l \in \partial i \setminus k} \psi_{l \rightarrow i}^s \right) \\ &\times \log \left[1 + e^{-\Delta S_{k \rightarrow i}} \prod_{l \in \partial i \setminus k} \psi_{l \rightarrow i}^s \right]. \end{aligned} \quad (8)$$

To obtain the entropy density of the network, we need to add the link between two randomly selected nodes and consider the entropy change due to this link addition. This includes two cases as depicted in figure 2 (a) and (b) respectively. In the first case where at most one end of the link takes positive cavity probability, then after the link addition, those configurations where both node i and j take the consumer state should be excluded from the ground state whose size is denoted by Ω , therefore, the number of the optimal assignments in the current ground state should be $\Omega' = \Omega - \Omega(1 - \psi_{i \rightarrow j}^s)(1 - \psi_{j \rightarrow i}^s)$ with the entropy change $\Delta S_{(ij)} = \log [1 - (1 - \psi_{i \rightarrow j}^s)(1 - \psi_{j \rightarrow i}^s)]$. In the second case where both ends of the added link are frozen into the consumer state before the link addition, the number of optimal assignments in the ground state without the link is $\Omega = \Omega_{N-2} \prod_{l \in \partial i \setminus j} \psi_{l \rightarrow i}^s \prod_{l' \in \partial j \setminus i} \psi_{l' \rightarrow j}^s = \Omega_{N-2} e^{\Delta S_{i \rightarrow j}} e^{\Delta S_{j \rightarrow i}}$ where Ω_{N-2} is the number of optimal assignments in the ground state without node i and j . After the link addition, either node i or node j changes its state to the source state. The number of optimal assignments in the current ground state becomes

$\Omega' = \Omega'|_{s_i=1} + \Omega'|_{s_j=1}$ where $\Omega'|_{s_i=1} = \Omega_{N-2} \prod_{l' \in \partial j \setminus i} \psi_{l' \rightarrow j}^s = \Omega_{N-2} e^{\Delta S_{j \rightarrow i}}$ and $\Omega'|_{s_j=1} = \Omega_{N-2} \prod_{l' \in \partial i \setminus j} \psi_{l' \rightarrow i}^s = \Omega_{N-2} e^{\Delta S_{i \rightarrow j}}$. Thus the entropy change due to the link addition in the second case is $\Delta S_{(ij)} = \log [e^{-\Delta S_{i \rightarrow j}} + e^{-\Delta S_{j \rightarrow i}}]$. To sum up, the entropy change due to the edge addition is written as

$$\begin{aligned} \Delta S_{(ij)} = & \Theta(\psi_{i \rightarrow j}^s + \psi_{j \rightarrow i}^s) \log [1 - (1 - \psi_{i \rightarrow j}^s)(1 - \psi_{j \rightarrow i}^s)] \\ & + [1 - \Theta(\psi_{i \rightarrow j}^s + \psi_{j \rightarrow i}^s)] \log [e^{-\Delta S_{i \rightarrow j}} + e^{-\Delta S_{j \rightarrow i}}]. \end{aligned} \quad (9)$$

The entropy density of source location problem in the transportation network can be evaluated in the Bethe approximation [3] through

$$s = \langle \Delta S_i \rangle - \frac{C}{2} \langle \Delta S_{(ij)} \rangle \quad (10)$$

where $\langle \cdot \rangle$ denotes both the disorder average and the average over the cavity message distribution. We evaluate the entropy density by population dynamics algorithm [1]. A population of \mathcal{N} pairs of $(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j})$ is used to approximate the joint distribution $P(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j})$ and its components are uniformly updated by the new computed ones according to Eq. (7). Usually, a number of iterations $\mathcal{T} - \mathcal{T}_0$ are used to compute the entropy value with \mathcal{T}_0 iterations for equilibration. Note that Eq. (10) gives a self-averaging entropy value in the thermodynamic limit in the sense that the typical value of the entropy computed by the population dynamics algorithm should be consistent with that computed on a single large network [3].

The source location problem has an interesting connection with the minimal vertex cover problem [13, 25] if we assume the source nodes as the covered nodes, and the consumer nodes as the uncovered nodes. The difference is, we introduce the surplus (frozen covered) nodes as the quenched disorder in the source location problem where the network is a random regular network with fixed node degree and the total energy cost consists of the installation cost for all converted source nodes and the transportation cost induced by the consumer nodes. This specific energy cost makes the reweighting factor defined in Sec. IIIB totally different from that in the minimal vertex cover problem.

B. One-step replica symmetry breaking analysis

When replica symmetry is broken, the single ground state would break up into exponentially many ground states plus an even larger set of metastable states acting as the dynamical traps for any greedy search algorithm. In this case, one should take into account the reshuffling of free energies of different states when cavity iterations are performed, therefore, we write the replicated free energy $\Phi(y)$ [26] as

$$e^{-yN\Phi(y)} \equiv \sum_{\alpha} e^{-yNf_{\alpha}} = \int df e^{N(\Sigma(f) - yf)} \quad (11)$$

where α indicates each state and $\Sigma(f)$ is the complexity function counting states with given free energy density f . A saddle point analysis of Eq. (11) gives $\Phi(y) = f^* - \Sigma(f^*)/y$ where f^* is determined by $y = d\Sigma(f)/df$. The inverse pseudotemperature y allows us to weight differently the various states according to their free energy densities while the usual inverse temperature $\beta = 1/T$ selects the energy of equilibrium configurations. Actually, Eq. (11)

corresponds to a decomposition of the Gibbs measure [27]. Here, our analysis is restricted to the ground state (β tends to infinity), with f tends to the energy density ϵ . For the current problem, the energetic complexity $\Sigma(\epsilon)$ can be computed by the following Legendre transform

$$\Sigma(\epsilon) = y(\epsilon - \Phi), \quad (12a)$$

$$\epsilon = \frac{\partial(y\Phi)}{\partial y}, \quad (12b)$$

where $\Sigma(\epsilon)$ can be computed from the parametric plot of $\epsilon(y)$ and $\Sigma(y)$ by varying the value of y . We remark here that the complexity function of the source location problem vanishes at finite y [7] and this was also observed in studies of the minimal vertex cover problem [25]. Therefore, the Parisi replica symmetry breaking parameter defined by $m = y/\beta$ vanishes in the zero temperature limit. The parameter m is used to select the size of the investigated ground state, as studied in Refs. [27–29] to compute the entropic complexity curve for random constraint satisfaction problems where the ground state energy can be set to zero and the replicated free energy Φ reaches the maximum at $y = \infty$ (in this case, $m \in [0, 1]$ [30]).

Due to the proliferation of pure states, we write the recursive equations for the joint distribution $P_{i \rightarrow j}(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j})$ at the 1RSB level as

$$\begin{aligned} P_{i \rightarrow j}(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j}) &= \mathcal{Z}_{i \rightarrow j}^{-1} \int \prod_{k \in \partial i \setminus j} dP_{k \rightarrow i}(\psi_{k \rightarrow i}^s, \Delta S_{k \rightarrow i}) e^{-y \Delta E_{i \rightarrow j}} \delta[\psi_{i \rightarrow j}^s - \mathcal{F}(\{\psi_{k \rightarrow i}^s, \Delta S_{k \rightarrow i}\})] \\ &\quad \times \delta[\Delta S_{i \rightarrow j} - \tilde{\mathcal{F}}(\{\psi_{k \rightarrow i}^s, \Delta S_{k \rightarrow i}\})] \end{aligned} \quad (13)$$

where $dP_{k \rightarrow i}(\psi_{k \rightarrow i}^s, \Delta S_{k \rightarrow i}) \equiv d\psi_{k \rightarrow i}^s d\Delta S_{k \rightarrow i} P_{k \rightarrow i}(\psi_{k \rightarrow i}^s, \Delta S_{k \rightarrow i})$. The function \mathcal{F} and $\tilde{\mathcal{F}}$ are given by Eq. (7a) and Eq. (7b) respectively. The reweighting factor $e^{-y \Delta E_{i \rightarrow j}}$ takes into account the energy change due to the addition of node i and its adjacent edges except (ij) . Note that $m = 0$ makes the contribution of the entropy change in this term disappear. The cavity energy values have been computed in Ref. [7]. It was found that the cavity energy $\Delta E_{i \rightarrow j}$ is equal to $u^2/2$ minus γ for every cavity consumer state in its neighborhood set $\partial i \setminus j$, where $\gamma \equiv u^2/2 - 1/(2C)$ which is non-negative in the singlet regime. Finally, $\mathcal{Z}_{i \rightarrow j}$ is a normalization constant. To simplify the analysis, we parameterize the joint distribution according to the discussion in Sec. III A as

$$P_{i \rightarrow j}(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j}) = p_{i \rightarrow j}^1 \delta(\psi_{i \rightarrow j}^s - 1) \delta(\Delta S_{i \rightarrow j}) + p_{i \rightarrow j}^0 \delta(\psi_{i \rightarrow j}^s) \tilde{\phi}_{i \rightarrow j}^0(\Delta S_{i \rightarrow j}) + p_{i \rightarrow j}^* \tilde{\phi}_{i \rightarrow j}^*(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j}) \quad (14)$$

where $\int d\tilde{\phi}_{i \rightarrow j}^0(\Delta S_{i \rightarrow j}) = 1$, $\int d\tilde{\phi}_{i \rightarrow j}^*(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j}) = 1$ and $p_{i \rightarrow j}^1 + p_{i \rightarrow j}^0 + p_{i \rightarrow j}^* = 1$. Compared with the RS case where the messages merely consist of the pair $(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j})$ for each directed edge, here the order parameter turns out to be a survey of these messages at the 1RSB level; $p_{i \rightarrow j}^1$, $p_{i \rightarrow j}^0$ and $p_{i \rightarrow j}^*$ tell us the probability of picking up a pure state at random and finding that the cavity state of node i is source, consumer and free respectively. These three surveys enable us to handle the state-to-state fluctuations at the 1RSB level. With this parametric representation, a finite y survey propagation equation can be obtained for

the source location problem as

$$p_{i \rightarrow j}^0 = \delta_{\Lambda_i, -1} \prod_{k \in \partial i \setminus j} \frac{p_{k \rightarrow i}^1 + p_{k \rightarrow i}^*}{p_{k \rightarrow i}^0 e^{y\gamma} + p_{k \rightarrow i}^1 + p_{k \rightarrow i}^*}, \quad (15a)$$

$$p_{i \rightarrow j}^* = \delta_{\Lambda_i, -1} \frac{\sum_{k \in \partial i \setminus j} p_{k \rightarrow i}^0 e^{y\gamma} \prod_{l \in \partial i \setminus j, k} [p_{l \rightarrow i}^1 + p_{l \rightarrow i}^*]}{\prod_{k \in \partial i \setminus j} [p_{k \rightarrow i}^0 e^{y\gamma} + p_{k \rightarrow i}^1 + p_{k \rightarrow i}^*]}, \quad (15b)$$

$$p_{i \rightarrow j}^1 = \delta_{\Lambda_i, A} + \delta_{\Lambda_i, -1} (1 - p_{i \rightarrow j}^0 - p_{i \rightarrow j}^*). \quad (15c)$$

After the fixed point of Eq. (15) is obtained, the replicated free energy can be computed via

$$\Phi(y) = \langle \Delta \Phi_i \rangle - \frac{C}{2} \langle \Delta \Phi_{(ij)} \rangle \quad (16)$$

where the average is taken over the capacity distribution and the survey distribution, and the replicated free energy shift $\Delta \Phi_i$ due to node addition (and its C edges) and $\Delta \Phi_{(ij)}$ due to link addition are, respectively,

$$\begin{aligned} -y \Delta \Phi_i &= \delta_{\Lambda_i, -1} \log \left[e^{-yu^2/2} \prod_{k \in \partial i} [p_{k \rightarrow i}^0 e^{y\gamma} + p_{k \rightarrow i}^1 + p_{k \rightarrow i}^*] + e^{-yu^2/2} (e^{y\gamma} - 1) \prod_{k \in \partial i} (p_{k \rightarrow i}^1 + p_{k \rightarrow i}^*) \right] \\ &\quad + \delta_{\Lambda_i, A} \log \prod_{k \in \partial i} [p_{k \rightarrow i}^0 e^{y\gamma} + p_{k \rightarrow i}^1 + p_{k \rightarrow i}^*], \end{aligned} \quad (17a)$$

$$-y \Delta \Phi_{(ij)} = \log \left[(p_{i \rightarrow j}^1 + p_{i \rightarrow j}^*)(p_{j \rightarrow i}^1 + p_{j \rightarrow i}^*) + [1 - (p_{i \rightarrow j}^1 + p_{i \rightarrow j}^*)(p_{j \rightarrow i}^1 + p_{j \rightarrow i}^*)] e^{y\gamma} \right], \quad (17b)$$

where we have used the energy changes calculated in Tables 5 and 6 in Ref. [7]. The energetic complexity can then be computed using Eq. (12). In the glassy phase, $\Sigma(\epsilon)$ typically increases from $y = 0$ up to the maximal point forming the first non-physical convex part, yet, with further increase in y , decreases down to the zero point where the complexity vanishes at the ground state energy ($y = y^*$). This second branch of the complexity curve is the physical concave part [2]. Actually, the zero complexity corresponds to the maximum of the replicated free energy since $\Sigma = y^2 \partial_y \Phi$ from the Legendre transform. To compute the ground state entropy at the 1RSB level, we should fix $y = y^*$.

To derive the formula for the ground state entropy at the 1RSB level, we first write the replicated free energy $-y\Phi(y, m) = \Sigma(\epsilon, s) - y\epsilon + ms$ keeping a finite value of m [28, 30] and at the end of the derivation, we get the ground state entropy via $s = \frac{\partial(-y\Phi(y, m))}{\partial m} \big|_{m=0}$ where $y = y^*$ determined by $\Sigma(y^*) = 0$. The 1RSB approximation of ground state entropy density

reads,

$$s = \left\langle \frac{[\Delta S_i e^{-y\Delta E_i}]}{[e^{-y\Delta E_i}]} \right\rangle - \frac{C}{2} \left\langle \frac{[\Delta S_{(ij)} e^{-y\Delta E_{(ij)}}]}{[e^{-y\Delta E_{(ij)}}]} \right\rangle, \quad (18a)$$

$$[\Delta S_i e^{-y\Delta E_i}] = \int \prod_{k \in \partial i} dP_{k \rightarrow i}(\psi_{k \rightarrow i}^s, \Delta S_{k \rightarrow i}) e^{-y\Delta E_i} \Delta S_i(\{\psi_{k \rightarrow i}^s, \Delta S_{k \rightarrow i}\}), \quad (18b)$$

$$[e^{-y\Delta E_i}] = \delta_{\Lambda_i, -1} e^{-yu^2/2} \left\{ \prod_{k \in \partial i} (p_{k \rightarrow i}^0 e^{y\gamma} + p_{k \rightarrow i}^1 + p_{k \rightarrow i}^*) + (e^{y\gamma} - 1) \prod_{k \in \partial i} (p_{k \rightarrow i}^1 + p_{k \rightarrow i}^*) \right\} \\ + \delta_{\Lambda_i, A} \prod_{k \in \partial i} (p_{k \rightarrow i}^0 e^{y\gamma} + p_{k \rightarrow i}^1 + p_{k \rightarrow i}^*), \quad (18c)$$

$$[\Delta S_{(ij)} e^{-y\Delta E_{(ij)}}] = \int dP_{i \rightarrow j}(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j}) dP_{j \rightarrow i}(\psi_{j \rightarrow i}^s, \Delta S_{j \rightarrow i}) e^{-y\Delta E_{(ij)}} \Delta S_{(ij)}, \quad (18d)$$

$$[e^{-y\Delta E_{(ij)}}] = (p_{i \rightarrow j}^1 + p_{i \rightarrow j}^*)(p_{j \rightarrow i}^1 + p_{j \rightarrow i}^*) + e^{y\gamma} \{1 - (p_{i \rightarrow j}^1 + p_{i \rightarrow j}^*)(p_{j \rightarrow i}^1 + p_{j \rightarrow i}^*)\}. \quad (18e)$$

Note that in the average $[\cdot]$, ΔS_i and $\Delta S_{(ij)}$ are still given by RS equations (8) and (9) respectively but the incoming messages to compute them should be sampled from the joint distribution Eq. (14) and the reweighting factor is also included. In addition, $\langle \cdot \rangle$ in the first term and second term in Eq. (18a) denote the averages over the capacity distribution and survey distribution and this average can be easily done by the population dynamics algorithm [1], which yields a typical value of the entropy density.

We evaluate the entropy numerically by population dynamics algorithm. At the 1RSB level, we require a population of \mathcal{N} pairs of $(p_{i \rightarrow j}^0, p_{i \rightarrow j}^*)$ with an additional population of \mathcal{M} pairs of $(\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j})$ associated with each element of the first population. Another relevant quantity is the fraction of source nodes f_s in the final optimized network. Its typical value can be evaluated as $f_s = \frac{1}{N} \sum_{i=1}^N \langle \psi_i^s \rangle$ at the RS level or 1RSB level. The average is taken over the disorder and the RS message or 1RSB survey distribution, and ψ_i^s is the full or marginal probability given by $\psi_i^s = \delta_{\Lambda_i, A} + \psi_i^s|_{\Lambda_i=-1}$ where $\psi_i^s|_{\Lambda_i=-1}$ is determined by Eq. (7a) in which $\partial i \setminus j$ is replaced by ∂i .

In the following studies, we choose $u = 2/3$ and $C = 3$ which fall within the range of the single regime $1/\sqrt{C} < u < \sqrt{(C+1)/[C(C-1)]}$ [7]. The entropy values of the ground state at both RS and 1RSB levels are given in figure 3. We use $(\mathcal{T}, \mathcal{T}_0) = (5000, 1000)$ for RS computation and $(\mathcal{T}, \mathcal{T}_0) = (1500, 500)$ for 1RSB computation. Only the mean value of the entropy density is plotted since the error bar is much smaller than the symbol size. In the RS population dynamics, the extremely small value for certain cavity probabilities can be observed when ϕ_s is sufficiently low, and this is equivalent to the divergence of the evanescent cavity fields [14, 31]. However, if we take a proper cutoff ε for the nearly vanishing probability, this situation can be circumvented [13] and the resulting entropy value seems to be insensitive to the cutoff value as long as $\varepsilon \geq 10^{-6}$ in the stable region of RS solution. For the 1RSB computation, we adopt $\varepsilon = 10^{-6}$ (smaller values of ε do not change much the result). In figure 3, the entropy density first increases when ϕ_s decreases, then reaches a maximum followed by the decreasing trend as the fraction of surplus nodes (disorder) further decreases. At a large value of ϕ_s , most of nodes are surplus nodes, the entropy should take a small value; when ϕ_s gets close to one, most of nodes are deficient nodes, which makes

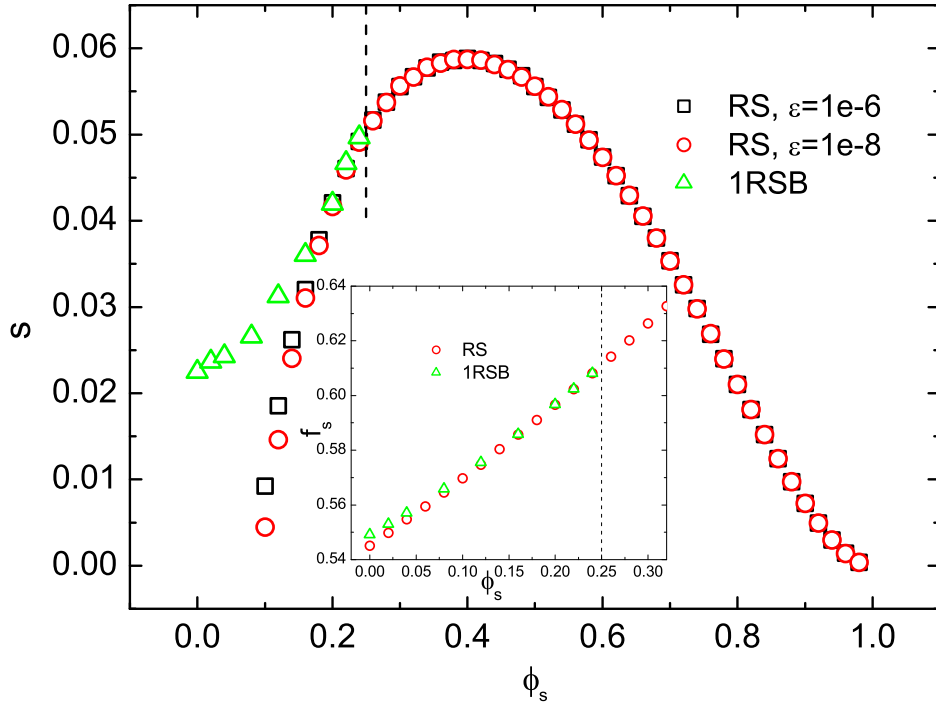


FIG. 3: (Color online) Ground state entropy density versus the fraction of surplus nodes ϕ_s . The vertical dashed line indicates the dynamical transition below which the RS description becomes incorrect. The RS results are obtained using different cutoffs $\varepsilon = 10^{-6}$ and 10^{-8} . The inset gives the typical ground state fraction of source nodes in the final optimized network. The network parameter $C = 3$ and the population size for RS computation is $\mathcal{N} = 10^5$, and $(\mathcal{N}, \mathcal{M}) = (2048, 1024)$ for 1RSB computation.

the source locations much more constrained and yields a small value of entropy. Due to the concavity of the entropy function, there should exist some maximum point and this point can be identified by the mean field computation in figure 3. The maximum implies that at the corresponding $\phi_s \simeq 0.40$, there exist a largest number of optimal assignments satisfying the singlet connection pattern among all values of ϕ_s .

At $\phi_s = 0.25$, the replica symmetry solution starts to be unstable against 1RSB perturbations [7]. This is confirmed by the fact that above this transition point, the complexity calculated by 1RSB equations (15) does not depend on the value of y and is always zero while it has a finite value below the transition point. This transition point is also called the dynamical transition threshold below which the local search process is usually trapped by the metastable states whose entropy density is called the complexity. For the source location problem, the replicated free energy reaches its maximum at finite y and actually the maximum corresponds to the ground state free energy [2]. In the zero temperature limit, one can thus obtain the ground state energy. Within the current context, the ground state entropy value under the 1RSB ansatz can also be computed improving the RS prediction.

We also show the fraction of source nodes in the ground state in the inset of figure 3 where the 1RSB ansatz predicts a higher value compared with the RS ansatz. We compute f_s through the linear relationship $f_s = (\epsilon - 1/(2C) + \phi_s u^2/2)/\gamma$ since we have the identity $(1 - f_s)/(2C) + (f_s - \phi_s)u^2/2 = \epsilon$. In this sense, the f_s serves as a measure of the ground state energy density. These values will be compared with those obtained from simulations on single instances in Sec. IV. As shown in the inset of figure 3, the f_s decreases as ϕ_s decreases. Actually, one deficient node causes an installation cost ($u^2/2$) when converted to a source node while it leads to an increment of $1/(2C)$ of the transportation cost when remaining as a consumer node. Notice that the consumer state is energetically favored since $\gamma > 0$ in the singlet regime. Consequently, on one hand, as ϕ_s decreases, the number of deficient nodes increases, on the other hand, to yield the minimal energy cost, some deficient nodes need to remain as consumer nodes while consumer nodes are not allowed to be paired in the singlet regime. The competition of these two effects leads to the decreasing trend of f_s with decreasing ϕ_s .

IV. NUMERICAL STUDIES ON SINGLE INSTANCES

In this section, we apply the entropic message passing approach derived in Sec. III A to study single instances of networks. Two kinds of decimation strategy are used to identify the optimal source location. One strategy is the maximal decimation, and the other is the reinforcement strategy. The belief propagation inspired decimation [7] is also compared. Hereafter, we use the short-hand notations BPD for belief propagation inspired decimation, EMPD for entropic message passing inspired decimation and EMPR for entropic message passing reinforcement algorithm.

A. Maximal decimation strategy

For the maximal decimation strategy, we fix a fraction f_d of N_t unfixed nodes with the largest full probability ψ_i^s to their most probable states once the algorithm converges at t -th sweep ($f_d = 2\%$ in our simulations). Each sweep consists of a sequential update of the messages on all the edges of the network in a random order. When $f_d N_t < 1$, only one node with the largest bias is fixed. In the glassy phase, the algorithm typically fails to converge, and this is due to the building up of the long range correlations of different parts of the network. Therefore, we carry out the decimation according to the average full probability (over a number of sweeps) instead of the instantaneous value of the probability [32]. When a node, say i is fixed, the network is simplified by the following decimation procedure. If the node is fixed to the source state, it will send out the message $\psi_{i \rightarrow j}^s = 1$ to all its neighbors $j \in \partial i$ regardless of the later recursions. Otherwise, all neighbors of node i should take source state and are fixed at the same time since no paired consumer nodes are allowed in the singlet regime. Correspondingly, these fixed source nodes send out a constant message $\psi_{i \rightarrow j}^s = 1$ to their neighbors. However, if at least one of neighbors of node i has been fixed to consumer in this case, a contradiction will be reported and we restart the algorithm. After the decimation, the recursion is carried out on the unfixed nodes of the network until $N_t = 0$ and an optimal assignment of source location is obtained. One can compute the marginal probability either based on the belief propagation $\psi_{i \rightarrow j}^s = \delta_{\Lambda_i, A} + \delta_{\Lambda_i, -1} \left[1 - \prod_{k \in \partial i \setminus j} \psi_{k \rightarrow i}^s \right]$

derived using the energetic cavity method [7], or according to the entropic message passing equations derived in Sec. III A. The pseudocode of EMPD is given as follows:

EMPD algorithm

INPUT: the network with a fraction ϕ_s of surplus nodes; a maximal number of iterations T_{max} ; a predefined precision η .

OUTPUT: one assignment in the singlet regime or 'probably no solutions'.

1. Initialize randomly $\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j}$ for all edges of the network except for the edges connected to the surplus nodes;
 2. for $t = 1$ to $t = T_{max}$ do:
 - 2.1 a sweep of random sequential update of the messages $\{\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j}\}$ for all edges according to Eq. (7);
 - 2.2 if $|\psi_{i \rightarrow j}^s(t) - \psi_{i \rightarrow j}^s(t-1)| < \eta$ and $|\Delta S_{i \rightarrow j}(t) - \Delta S_{i \rightarrow j}(t-1)| < \eta$ for all edges and the iteration has converged: goto line 3;
 3. if $t < T_{max}$, compute the full probability ψ_i^s for unfixed nodes and decimate the network, else do the time average of the full probability over the later $T_{max}/2$ sweeps and decimate the network;
 4. if an optimal assignment is found, return the solution and stop; else if no contradiction is found, continue the decimation procedure on the smaller network (goto line 2); else if a contradiction is met, return 'probably no solutions' and stop.
-

B. Soft decimation strategy

The reinforcement strategy has been used to find solutions for random constraint satisfaction problems [19–21]. Here, we introduce an external bias μ_i for each node, and its intensity is updated with a probability increasing with the running time as $1 - t^{-r}$. The external bias μ_i is updated as $\mu_i = \pi$ if $\psi_i^s < 0.5$ and $1 - \pi$ otherwise. The bias strength $\pi \in [0, 0.5]$. This implies that the current value of the cavity probability of one node will be enhanced by a factor if its full probability at the preceding iteration is biased towards the source state. Thus the only modification to the original recursive equation Eq. (7) is

$$\tilde{\psi}_{i \rightarrow j}^s = \frac{\mu_i \psi_{i \rightarrow j}^s}{\mu_i \psi_{i \rightarrow j}^s + (1 - \mu_i)(1 - \psi_{i \rightarrow j}^s)} \quad (19)$$

where $\tilde{\psi}_{i \rightarrow j}^s$ is the reinforced cavity probability while $\psi_{i \rightarrow j}^s$ is the original one computed from Eq. (7). Both π and r should be optimized so as to properly guide the iteration to converge to an optimal assignment. In the numerical simulation, we fix $r = 0.1$ although other choices are also able to find a solution, e.g., $r = 0.3$. The algorithm is precisely described as follows:

EMPR algorithm

INPUT: the network with a fraction ϕ_s of surplus nodes; a maximal number of iterations T_{max} ; two empirical parameters π and r .

OUTPUT: one assignment in the singlet regime or 'probably no solutions'.

1. Initialize randomly $\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j}$ for all edges of the network except for the edges connected to the surplus nodes and initialize the external biases $\{\mu_i\}$ at random;
 2. for $t = 1$ to $t = T_{max}$ do:
 - 2.1 a sweep of random sequential update of the messages $\{\psi_{i \rightarrow j}^s, \Delta S_{i \rightarrow j}\}$ for all edges according to Eq. (7) and Eq. (19);
 - 2.2 update all the external biases with probability $1 - t^{-r}$;
 - 2.3 assign $s_i = 1$ if $\mu_i > 0.5$ and 0 otherwise; if $\{s_i\}$ is an optimal assignment (no contradiction), return the solution and stop.
 3. if $t = T_{max}$, return 'no solutions at current values of π and r '.
-

The inference results using the above proposed algorithms on single instances of size $N = 2000$ are compared in figure 4. As expected, the entropic message passing algorithm yields a lower value of f_s than the belief propagation without taking the entropic effects into account. It should be mentioned that EMPR can find the optimal assignments faster than other strategies in the glassy regime. This is because, on one hand, the presence of an updating external bias manages to drive the evolution of the reinforced cavity probabilities towards the ground state where the satisfying assignment is maximally aligned with the externally imposed direction; on the other hand, the hard decimation fails to converge in this regime and the time-average of the marginal probabilities is required, which increases the time complexity of the algorithm. However, in the RS region, EMPR gets slow because the proper values of (π, r) could not be easily found, probably due to the presence of the many background messages sent out by the surplus nodes. On the other hand, the EMPD and BPD are typically convergent in this regime and thus fast to identify the optimal assignment. In the glassy regime, the estimated f_s by message passing algorithms is higher than the ground state value predicted by the 1RSB theory. The physical interpretation is, in the glassy region, exponentially many metastable states act as dynamical arrests for various simple heuristics. Furthermore, the ground state predicted by the 1RSB theory can be unstable towards further steps of RSB or full RSB (an infinite hierarchy of nested states) [3, 7], and the lower bound to the true dynamical threshold of f_s is predicted by the Gardner value [33] above which the 1RSB metastable states become unstable [34]. However, the entropic message passing algorithm on single instances yields a lower value of f_s , and particularly the EMPD can give a lowest value of f_s among all message passing inspired algorithms in the glassy region.

V. CONCLUSION

We have considered the entropic effects in the ground state of the source location problem and derived the associated entropic message passing algorithms as an improvement over the

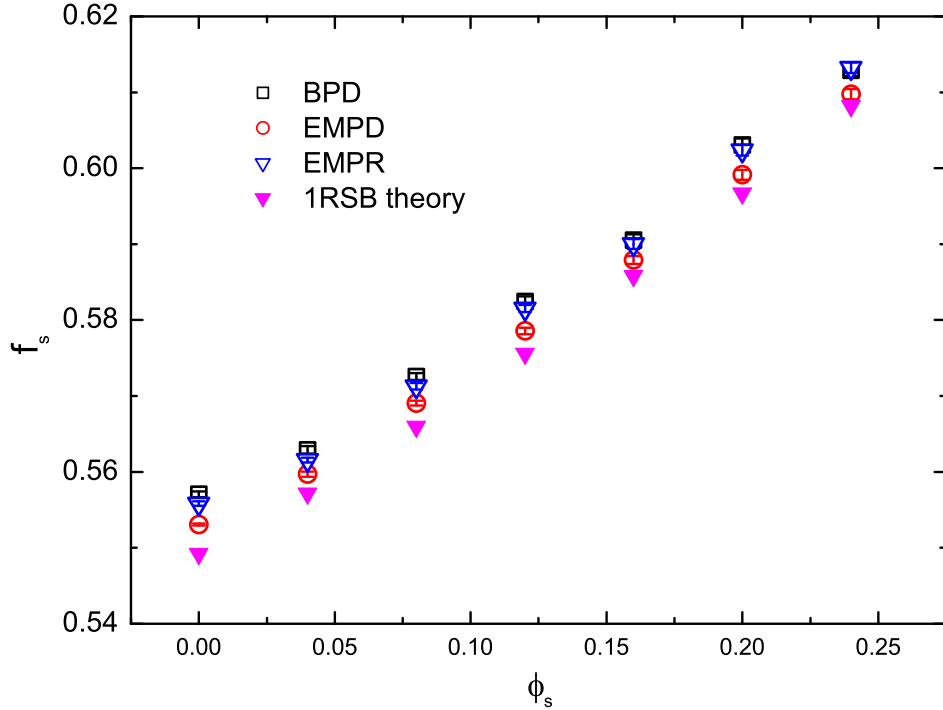


FIG. 4: (Color online) The fraction of source nodes in the optimized network ($N = 2000$) obtained by different algorithms. For BPD and EMPD, $T_{max} = 2000$ and $\eta = 10^{-3}$ and additionally for EMPD, $\varepsilon = 10^{-6}$. $T_{max} = 10^4$ for EMPR. The data point is the mean over 30 random regular networks with $C = 3$ and the error bar is also shown. The entropic message passing algorithms show lower values of f_s close to the ground state values predicted by the 1RSB theory in the glassy regime.

previous energetic computation of the problem. We assume that the cavity iterations will affect the size of the ground state and a self-consistent relation between a pair of messages (cavity probability and cavity entropy) can be obtained in a probabilistic way. Using the one-step replica symmetry breaking ansatz, the ground state entropy predicted by the RS solution is improved and yields a better approximation. Additionally, the predicted f_s are checked by the simulations on single instances of transportation networks. Using the message passing inspired decimation algorithms, we find that the entropic information helps to lower the inference value of f_s making it closer to the ground state value.

As the installation cost parameter u increases, another important configuration of source and consumer nodes, namely the doublet regime will appear. Extension of the current method to this regime would be very interesting. In this case, to derive the recursive equations, we need to define two extra cavity messages, one for singly consuming state and the other for doubly consuming state. We leave this for future work. It is also of interest to extend the current analysis to the lattice glass models [35, 36] where for example each site in the (finite dimensional or Bethe) lattice can have at most one particle and any particle

has at most a fixed number of occupied nearest neighbor. Other possible applications may be found in routing and path selection problems on sparse graphs [8].

Acknowledgments

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